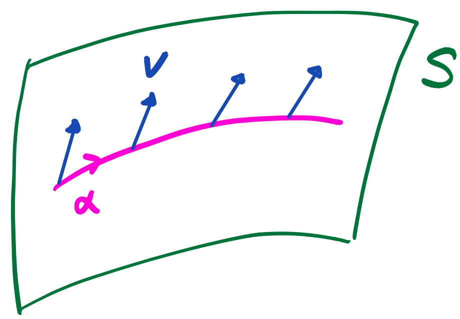
§ Parallel Transport

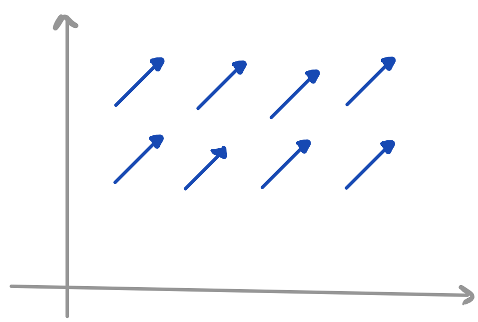
Let $\alpha : [a, b] \longrightarrow S$ be a curve on S.

Suppose V is a tangential vector field defined on α.



Question: When do we consider V as "parallel" along α?

Note: The concept of "parallel" is clear in $\mathbb{R}^n$ since we can translate vectors from a point to any other point.



a parallel vector field in $\mathbb{R}^2$

BUT there is no "translations" on a surface S!

So we need to make a definition:

Def$^{\underline{n}}$: A tangential vector field V is parallel along a curve $\alpha : [a,b] \to S$ on a surface S if

$$\boxed{\nabla_{\alpha'} V \equiv 0}$$

i.e. V is constant as seen intrinsically on the surface.

Prop: If V_1, V_2 are two parallel tangential vector fields along a curve α on S, then

$$\boxed{\langle V_1, V_2 \rangle \equiv \text{constant}}$$

Proof: Let $\alpha(t) : [a,b] \to S$. Then we can think of $\langle V_1, V_2 \rangle (t) = \langle V_1(\alpha(t)), V_2(\alpha(t)) \rangle$ as a function of t. Using metric compatibility of ∇

$$\frac{d}{dt} \langle V_1, V_2 \rangle = \langle \underbrace{\nabla_{\alpha'} V_1}_{= 0}, V_2 \rangle + \langle V_1, \underbrace{\nabla_{\alpha'} V_2}_{= 0} \rangle = 0$$

$\square$

Cor: (1) A parallel vector field has constant length.

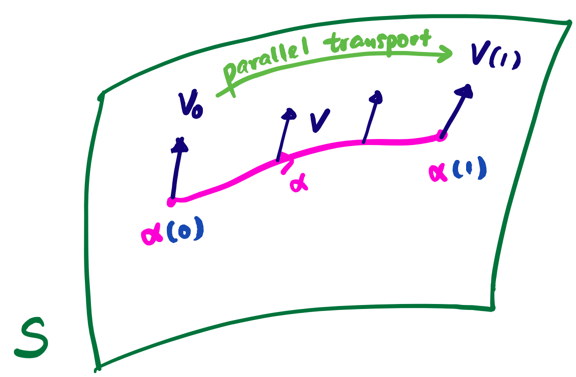
(2) Two parallel vector fields have constant angle between them.

Now, having the concept of "parallelism" along curves, we can move a vector along a given curve in such a way that the vector is "unchanged" as seen on S.

Thm: Let $\alpha : [0,1] \to S$ be a curve on S.
For any given $V_0 \in T_{\alpha(0)}S$, there exists a unique parallel tangential vector field V defined along α s.t.

$$V(0) = V_0 .$$

Note: The vector $V(1) \in T_{\alpha(1)}S$ is said to be the parallel transport of V_0 from $\alpha(0)$ to $\alpha(1)$ along the curve α.



$$\nabla_{\alpha'} V \equiv 0$$

Proof: We first express the parallel condition $\nabla_{\alpha'} V \equiv 0$ in local coordinates (u^1, u^2). Let $\alpha(t) = (u^1(t), u^2(t))$.

$$\alpha'(t) = \frac{du^1}{dt}\partial_1 + \frac{du^2}{dt}\partial_2$$

$$V(t) = v^1(t)\,\partial_1 + v^2(t)\,\partial_2$$

and $\qquad \nabla_{\partial_i}\partial_j = T_{ij}^k\,\partial_k$.

Therefore, we have

$$\begin{aligned}
\nabla_{\alpha'} V &= \nabla_{\frac{du^i}{dt}\partial_i}(v^j \partial_j) \\
&= \frac{du^i}{dt}(\partial_i v^j)\,\partial_j + \frac{du^i}{dt} v^j\, \nabla_{\partial_i}\partial_j \\
&= \frac{du^i}{dt}(\partial_i v^j)\,\partial_j + \frac{du^i}{dt} v^j\, T_{ij}^k\,\partial_k \\
&= \left(\frac{d}{dt} v^k + \frac{du^i}{dt} T_{ij}^k v^j \right) \partial_k
\end{aligned}$$

Hence, $\nabla_{\alpha'} V \equiv 0$ is equivalent to the following linear 1st order system of ODEs for the unknowns v^1, v^2:

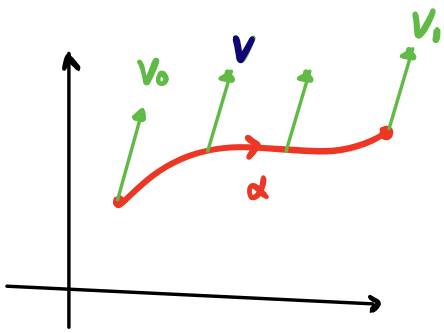
$$(*) \quad \boxed{\frac{dv^k}{dt} + \frac{du^i}{dt} T_{ij}^k v^j = 0} \qquad (k = 1, 2)$$

which is uniquely solvable given $v^1(0), v^2(0)$.

$\blacksquare$

Examples:

(1) Plane

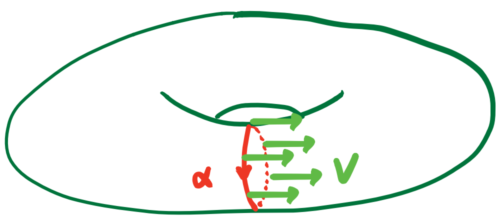


The parallel vector field is

$$V \equiv V_0$$

(2) Torus of revolution

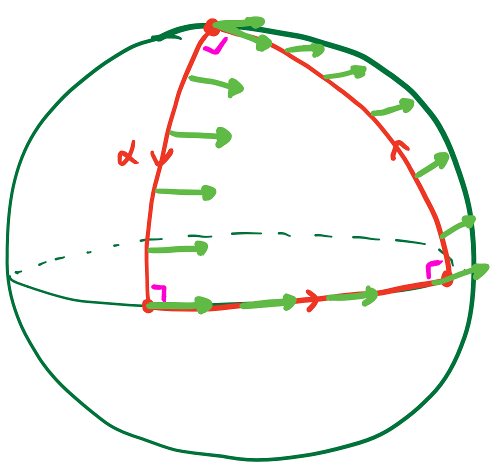
α : a meridian



Note: Parallel transport of a vector around the loop α returns to the same vector.

(Not true in general!)

(3) Sphere



Note: Parallel transport of a vector around this closed loop α gives a different vector than the vector we begin with!

§ Geodesics

Def$^{\underline{n}}$: A curve $\alpha : [a,b] \to S$ is said to be a geodesic on the surface S if

$$\boxed{\nabla_{\alpha'} \alpha' \equiv 0}$$

Note: In other words, the tangent vector field α' is parallel along α. From our discussion above,

$$\| \alpha' \| \equiv \text{constant}$$

i.e. any geodesic α is automatically parametrized proportional to arc length.

Prop: α is geodesic if and only if in any local coordinate system $\alpha(t) = (u^1(t), u^2(t))$, we have

$$(\#) \quad \boxed{\frac{d^2 u^k}{dt^2} + \Gamma_{ij}^k(\alpha(t)) \frac{du^i}{dt} \frac{du^j}{dt} = 0} \qquad k = 1,2$$

Proof: Plug $v^i = \frac{du^i}{dt}$ into $(*)$.

Remark: $(\#)$ is a system of 2^{nd} order, non-linear ODEs.

By standard ODE theory, we have the following:

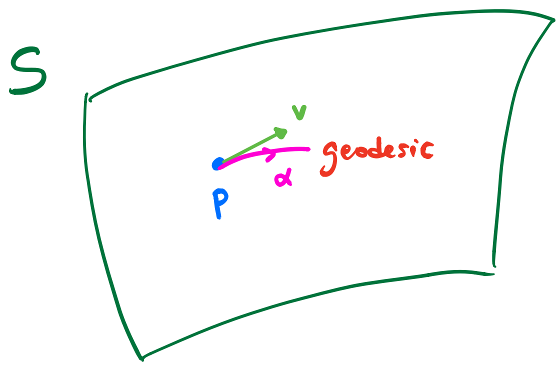
Theorem: Let $S \subseteq \mathbb{R}^3$ be a surface.

For any given $p \in S$ and $v \in T_pS$,

there exists $\varepsilon > 0$ and a unique geodesic

$$\alpha : [0, \varepsilon) \longrightarrow S$$

s.t. $\alpha(0) = p$, $\alpha'(0) = v$.



"Short time existence & uniqueness for geodesics"

As an example, we "compute" the geodesics lying on a plane in two different coordinate systems:

① Rectangular coordinates

$X_{rec}(x, y) = (x, y, 0)$

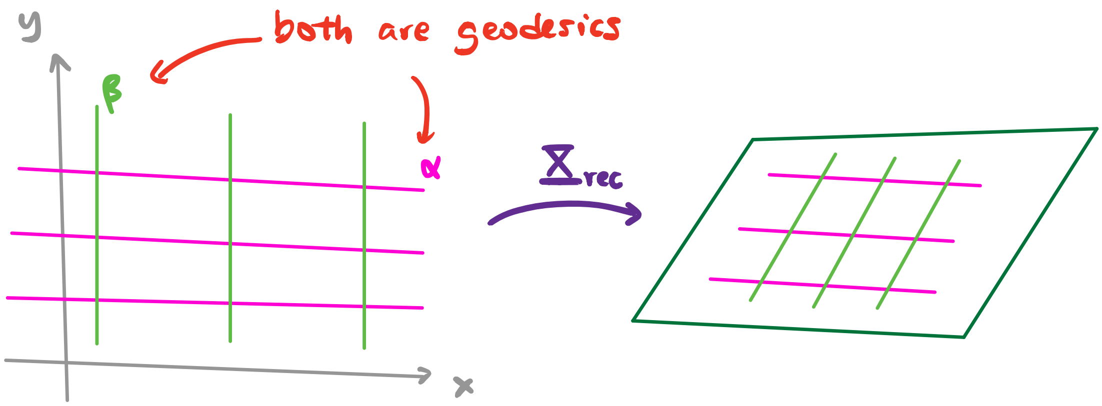
$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Gamma_{ij}^{k} = 0 \quad \forall\, i, j, k$

(#) becomes:

$$\begin{cases} \dfrac{d^2x}{dt^2} = 0 \\[2ex] \dfrac{d^2y}{dt^2} = 0 \end{cases} \Rightarrow$$

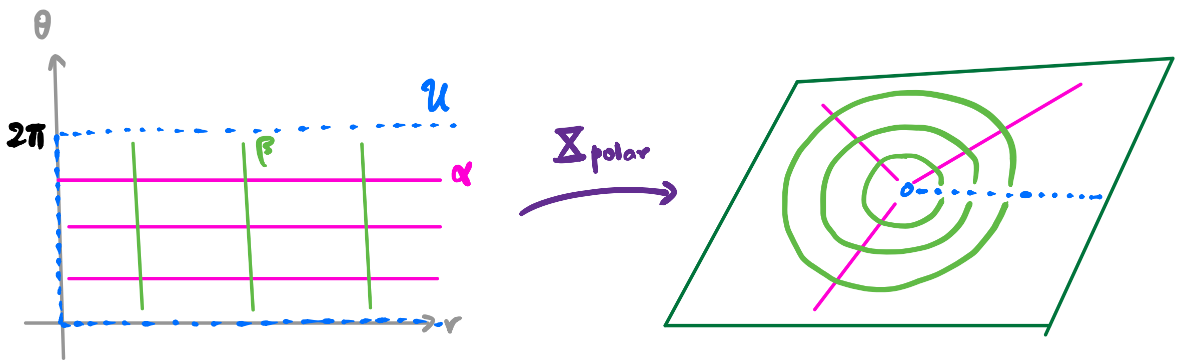
$x(t)$, $y(t)$ are linear functions of t.



② Polar coordinates

$$\mathbf{X}_{polar}(r, \theta) = (r\cos\theta, r\sin\theta, 0)$$

where $r > 0$, $0 < \theta < 2\pi$



$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

$$\begin{cases} \Gamma^r_{rr} = \Gamma^r_{r\theta} = \Gamma^\theta_{rr} = \Gamma^\theta_{\theta\theta} = 0 \\ \Gamma^r_{\theta\theta} = -r \\ \Gamma^\theta_{r\theta} = \frac{1}{r} \end{cases}$$

(#) becomes:

$$\boxed{\begin{aligned} r'' - r(\theta')^2 &= 0 \\ \theta'' + \frac{2}{r} r'\theta' &= 0 \end{aligned}} \quad (\star)$$

α : $\theta \equiv \text{const.}$, $r(t) = At + B$ solves $(\circledast)$

hence they are geodesics!

β : $r \equiv \text{const.}$, $\theta(t) = At + B$ does **NOT** solve $(\circledast)$

(unless $A = 0$, degenerate!)

hence they are **NOT** geodesics!